

Once again, the Schrodinger Eq'n:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \underbrace{-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V \Psi(x,t)}_{\hat{H} \Psi(x,t)}$$

(which can also be written as $\hat{H} \Psi(x,t)$ if you like.)

And once again, assume $V = V(x)$ (no t in there!)

We can start to solve this PDE by

SEPARATION OF VARIABLES

Assume (hope? wonder ~~ed~~ if?) we might find a sol'n of form $\Psi(x,t) = u(x) \phi(t)$

(Griff calls $u(x) \equiv \Psi(x)$, but I can't distinguish a "small Ψ " from the "capital Ψ " so easily in my handwriting)

so $\frac{\partial \Psi}{\partial t} = u(x) \frac{d\phi}{dt}$ ← full derivative!

$$\frac{\partial^2 \Psi}{\partial x^2} = \phi(t) \frac{d^2 u(x)}{dx^2}$$

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So Schrod eq'n reads (with $\frac{d\phi}{dt} \equiv \dot{\phi}$, and $\frac{du}{dk} \equiv u'$)

$$i\hbar(u \cdot \dot{\phi}) = -\frac{\hbar^2}{2m}(\phi u'') + V(u \cdot \phi)$$

Now divide both sides by $\Psi = u \cdot \phi$

$$\underbrace{i\hbar \frac{\dot{\phi}(t)}{\phi(t)}}_{\text{function of time only}} = \underbrace{-\frac{\hbar^2}{2m} \frac{u''(x)}{u(x)} + V(x)}_{\text{function of space only}}$$

function of
time only

function of
space only.

This is not possible unless both sides are constants.

Convince yourself, that is the key to "method of separation of variables".

Let's name this constant "E"

So (1) $i\hbar \dot{\phi}(t) = E \phi(t)$

← ordinary
O.D.E.'s
✓

(2) $-\frac{\hbar^2}{2m} u''(x) + V(x) u(x) = E u(x)$

[Note units of E are $\frac{\hbar^2}{2m \cdot (\text{dist})^2}$ or $\frac{\hbar}{\text{time}}$, either way, check, it's Energy!]

Eq'n 1 is about as simple as ODE's get!

check $\varphi(t) = \varphi_0 e^{-iEt/\hbar}$
 \hookrightarrow any constant, it's linear ODE.

(1st order linear ODE is supposed to give one undetermined constant, right?)

This is "universal", no matter what $V(x)$ is,
 once we find a $u(x)$, we'll have a corresponding
 $\Psi(x,t) = u(x) \varphi(t) = u(x) e^{-iEt/\hbar}$

But be careful, that $u(x)$ depends on the E ,

$$(2) \quad -\frac{\hbar^2}{2m} u''(x) + V(x) u(x) = E u(x).$$

\uparrow
 this is $\frac{d^2 u(x)}{dx^2}$. This is the "time independent Schrodinger Equation".

you can also write it

$$\hat{H} [u(x)] = E \cdot u(x) \quad \leftarrow \text{an "eigen value equation"}$$

$$\hat{H} = \text{"Hamiltonian" operator} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} [] + V(x)$$

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In general, $\hat{H}[u(x)] = E u(x)$ has many possible solns!

\nearrow eigenfunctions \uparrow eigenvalues

$u_1(x), u_2(x), \dots, u_n(x)$ may all work, each corresponding to some particular eigenvalue E_1, E_2, \dots, E_n .

(what we will find is not any old E is possible if you want u to be normalizable & well behaved. Only certain ~~old~~ E 's, the E_n 's, will be OK.)

Such a $u_n(x)$ is called a "stationary state of \hat{H} ".

why?

well, notice that $\Psi_n(x, t)$ corresponding to u_n is

$$\Psi_n(x, t) = u_n(x) e^{-i E_n t / \hbar} \quad \leftarrow \text{go back a page!}$$

so $|\Psi_n(x, t)|^2 = |u_n(x)|^2 \quad \leftarrow \text{no time dependence}$

(for the probability density) It's not evolving in time, it's "stationary". (Because $|e^{-i E_n t / \hbar}|^2 = 1$)
convince yourself!

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(If you think back to de Broglie's free particle

$$\Psi_{\text{free}} \approx A e^{iKx} e^{-i\omega t}, \text{ with } E = \hbar \omega,$$

it looks like we have stationary states $\Psi_n(x, t)$

with this (same) simple time dependence, $e^{i\omega_n t}$,

with $\omega_n = E_n / \hbar$.)

If you compute $\langle \hat{Q} \rangle$ for a stationary state,

the $e^{iE_n t / \hbar}$ in Ψ multiplies $e^{-iE_n t / \hbar}$ in Ψ^* ,

$$\langle \hat{Q} \rangle = \int \Psi_n^*(x, t) \hat{Q}(x, \frac{\hbar}{i} \frac{\partial}{\partial x}) \Psi_n(x, t) dx$$

in state
" Ψ_n "

again, no time dependence.

Stationary states are dull, nothing about them

(that's measurable) changes w. time. (Hence, "stationary")

→ (Not all states are stationary... just these special states, the Ψ_n 's)

And remember, $\hat{H} u_n(x) = E_n u_n(x) \leftarrow \begin{matrix} \text{Time Indep.} \\ \text{Schrod eq'n} \\ (2) \end{matrix}$

and also $\hat{H} \Psi_n(x, t) = E_n \Psi_n(x, t)$

This is original Schrod Eq'n! (after plugging in our time sol'n for $\phi_n(t)$.)

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check this out, though: for state $\bar{\Psi}_n(x,t)$

$$\langle \hat{H} \rangle = \int \bar{\Psi}_n^* \hat{H} \Psi_n dx = \int \bar{\Psi}_n^* E_n \Psi_n dx$$

in stationary

$$\bar{\Psi}_n = E_n \int \bar{\Psi}_n^* \Psi_n dx = E_n$$

↑
constants come out
of integrals

this is normalization!

so state $\bar{\Psi}_n$ "has energy eigenvalue E_n "

and has expectation value of energy E_n

$$\begin{aligned} \underline{\text{and}} \quad \langle \hat{H}^2 \rangle &= \int \bar{\Psi}_n \hat{H}^2 \Psi_n = \int \bar{\Psi}_n \hat{H} (\hat{H} \bar{\Psi}_n) dx \\ &= \int \bar{\Psi}_n^* \hat{H} (E_n \bar{\Psi}_n) = E_n \int \bar{\Psi}_n^* \hat{H} \bar{\Psi}_n dx \\ &= E_n^2 \end{aligned}$$

given E_n !

$$\text{so } \sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = E_n^2 - E_n^2 = 0$$

CONCLUSION:

$\bar{\Psi}_n$ is a state with a definite energy E_n .

(no uncertainty!)

(that's why we picked the letter E for this eigenvalue!)

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Remember, $\hat{H} \Psi = i\hbar \frac{\partial \Psi}{\partial t}$ is linear, so

If Ψ_1 & Ψ_2 are sol'n's, so is $(a\Psi_1 + b\Psi_2)$

But, this linear combo is not stationary! It does not have an "energy eigenvalue" associated with it!

Most general sol'n of $\hat{H} \Psi = i\hbar \frac{\partial \Psi}{\partial t}$ is

$$\Psi_{\text{general}}(x, t) = \sum_{n=1}^{\infty} C_n \Psi_n(x, t) = \sum_{n=1}^{\infty} C_n U_n(x) e^{-iE_n t / \hbar}$$

↑
not an energy eigenstate
not stationary
 It's a "mixed energy"!

↑
 Energy eigenstate, or
 "Stationary state"

Simple
 time
 dependence
 for each
 term

any constants you like, real or complex,

Just so long as you ensure Ψ is normalized.

Any/
Every physical state can be expressed in this way, as a
 combination of the more special stationary states, or "eigenstates"

If you measure energy on a stationary state, you
 get E_n , (definite value)

(But if you measure energy on a mixed or general state,
~~you~~ we need to discuss this further! Hang on...